Analytic Continuation and Real-Time

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with Andrei Alexandru and Paulo Bedaque

Based on 19xx.xxxxx

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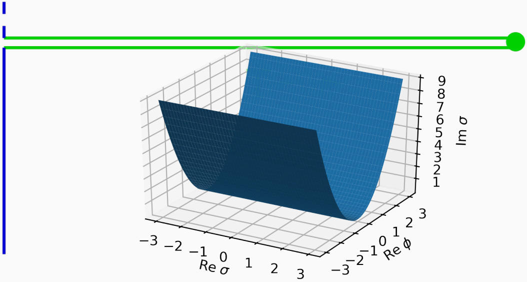
SIGN'19 på Syddansk Universitet



Real-Time Correlators

Euclidean lattice field theory makes equilibrium physics “easy”.

$$\langle \mathcal{O} \rangle = Z^{-1} \operatorname{Tr} e^{-\beta H} \mathcal{O}$$

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$$\langle \mathcal{O}(t)\mathcal{O}(0) \rangle = Z^{-1} \operatorname{Tr} e^{-\beta H} \mathcal{O}(t)\mathcal{O}(0)$$

Alternatively, evolve with a different Hamiltonian entirely.

$$\langle \mathcal{O} \rangle (t) = \operatorname{Tr} e^{-\beta H} e^{i\tilde{H}t} \mathcal{O} e^{-i\tilde{H}t}$$

Who Cares?

Green-Kubo relations

Real-time separated correlators $\Rightarrow$ viscosity, conductivity...

$$\eta = \int_0^\infty \mathrm{d}t \int_V \mathrm{d}^3 r \ \langle \pi_{ij}(r,t)\pi_{ij}(0,0)\rangle$$
$$\sigma = \int_0^\infty \mathrm{d}t \ \langle j(t) j(0)\rangle$$
$$D = \int_0^\infty \mathrm{d}t \ \langle v(t) v(0)\rangle$$

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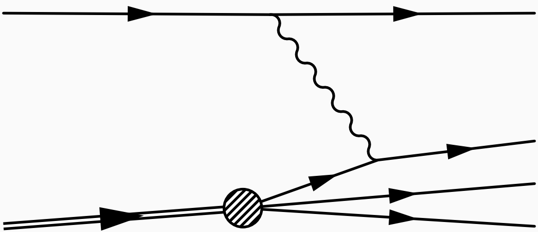
- Heavy ion physics
- KSS Conjecture ($\frac{\eta}{s} \leq \frac{1}{4\pi}$)

Who Cares?

Hadronic Physics

- Parton Distribution Functions

$$f(x) \sim \int e^{ixP^+ z} \langle P | \bar{\psi}(x,t) W \psi(0,0) | P \rangle$$



- Hadronic Tensor

$$W_{\mu\nu} \sim \int \mathrm{d}^3x \, \mathrm{d}t \, e^{i\vec{k}x} e^{ik^0 t} \langle P | J_\mu(x,t) J_\nu(0,0) | P \rangle$$

The Competition

Transport Coefficients

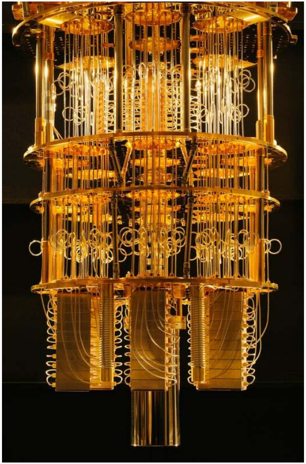
- Fitting the spectral function
- Backus-Gilbert method

Hadronic Physics

- Quasi-PDFs
- Maximum entropy method for analytic continuation

Get Euclidean correlator and analytically continue.

The Competition



50 qubit machines now available.

Qubits decohere too quickly to be useful.

Currently limited to $\lessapprox 10$ qubits in practice.

Schwinger-Keldysh Action

$$Z = \operatorname{Tr} e^{-\beta H} = \operatorname{Tr} e^{-\delta T} e^{-\delta K} e^{-\delta T} \cdots$$

$$Z = \int \mathrm{d}x_1 \cdots \langle x_1 | \, e^{-\delta V} e^{-\delta T} \, | x_2 \rangle \cdots$$

$$Z = \int \mathrm{d}x_1 \cdots e^{-S(x_1, \ldots)}$$

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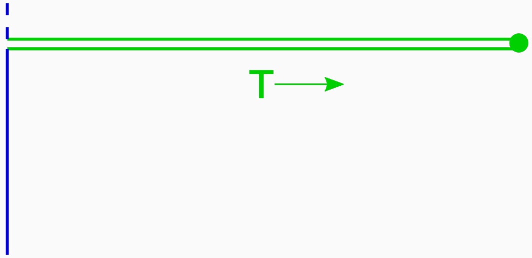

Real-Time Sign Problems

$$Z = \int \mathcal{D}\phi \; e^{-S[\phi]}$$

Sample according to $\operatorname{Re} S$; reweight with $\operatorname{Im} S$.

$$\langle \mathcal{O} \rangle = \underbrace{\frac{\int \mathcal{D}\phi \; \mathcal{O} e^{-\operatorname{Im} S[\phi]} e^{-\operatorname{Re} S[\phi]} \Big/ \int \mathcal{D}\phi \; e^{-\operatorname{Re} S[\phi]}}{\int \mathcal{D}\phi \; e^{-\operatorname{Im} S[\phi]} e^{-\operatorname{Re} S[\phi]} \Big/ \int \mathcal{D}\phi \; e^{-\operatorname{Re} S[\phi]}}}_{\langle \sigma \rangle}$$

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$$S = V(\phi_1) + \cdots + iV(\phi_i) + \cdots$$

So, $\operatorname{Re} S$ is independent of ϕ_i.

There are flat directions!

Real-Time Sign Problems

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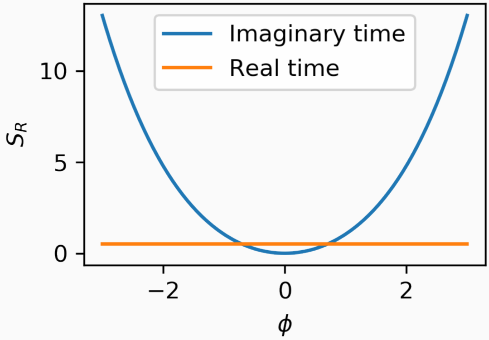

This sign problem is "infinitely bad".

This is not generic

Linear sigma model: $U(1)$ fields.

$$S_E = \frac{1}{2}\sum_x (1 - n_x \cdot n_{x+1})$$

The domain of integration is compact.

$$Z_Q = \int_{S^1} \mathrm{d}n_1 \int_{S^1} \mathrm{d}n_2 \cdots e^{-\operatorname{Re} S}$$

Average sign strictly greater than 0: $\langle \sigma \rangle > 0$.

A Toy Sign Problem

$$Z = \int \mathrm{d}x \; e^{ix^2 + i\lambda x^4}$$

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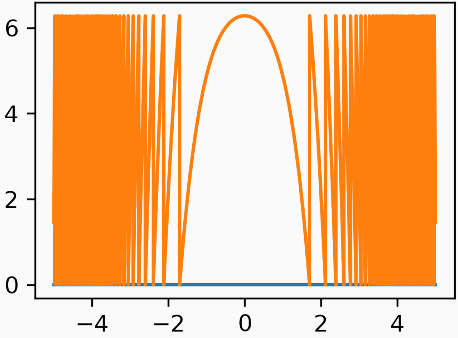

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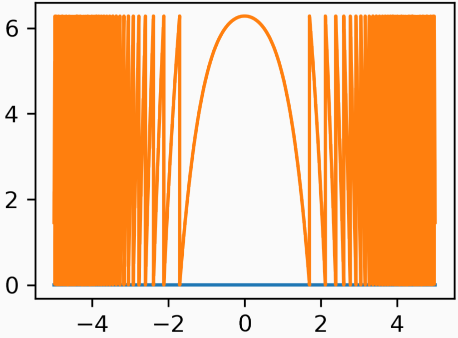

Monte Carlo will not converge.

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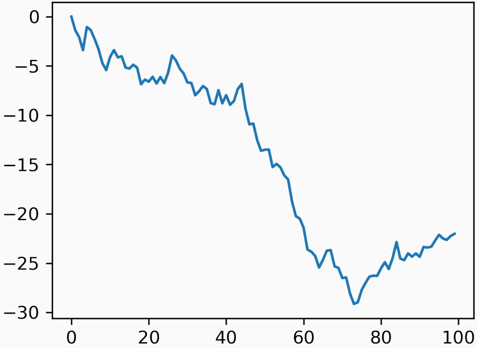

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Gaussians integrals

We know how to take Gaussian integrals. No matter how bad the sign problem is!

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This isn't a new idea.

This isn't new...

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$$S = \int \mathrm{d}^2x \, \bar{\psi} \left(\partial\!\!\!/ - m \right) \psi - g^2 \left(\bar{\psi}\psi \right)^2$$

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$$S = \int \mathrm{d}^2 x \frac{1}{g^2} \sigma^2 + \bar{\psi} \left(\partial\!\!\!/ - m - 2\sigma \right) \psi$$

Hubbard-Stratonovich transformation / "Completing the square"

Hubbard-Stratonovich for Scalars

Here's our ugly sign problem:

$$Z = \int \mathrm{d}x \; e^{ix^2 + i\lambda x^4}$$

Introduce an auxiliary 'field'.

$$Z = \int \mathrm{d}x\mathrm{d}\sigma \; e^{ix^2 + i\lambda x^4} e^{-\sigma^2}$$

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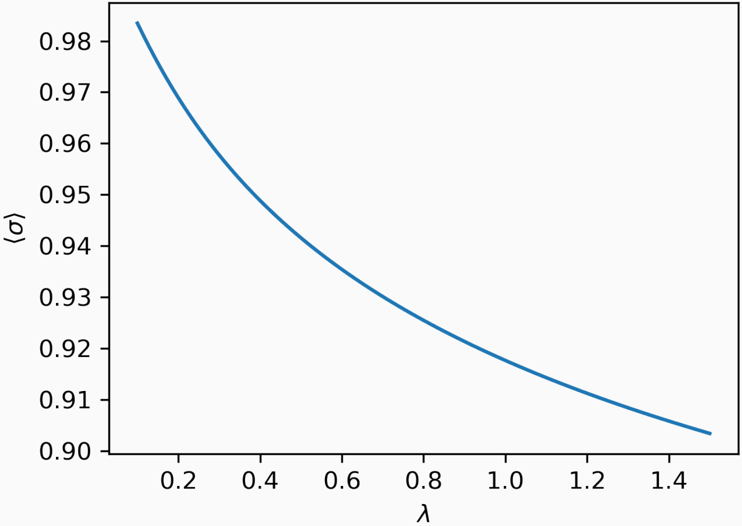
And perform a Gaussian integral!

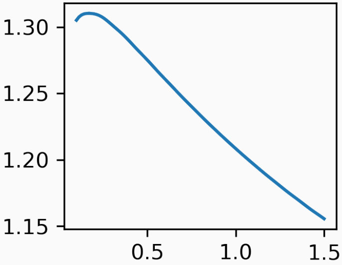
$$Z = \int \mathrm{d}x\mathrm{d}\sigma \; e^{-\sigma^2} \frac{1}{\sqrt{i + 2i\sqrt{i\lambda}\sigma}}$$

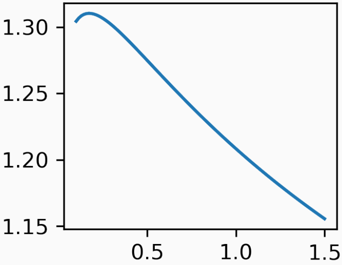
A Finite Sign Problem



Sanity Check

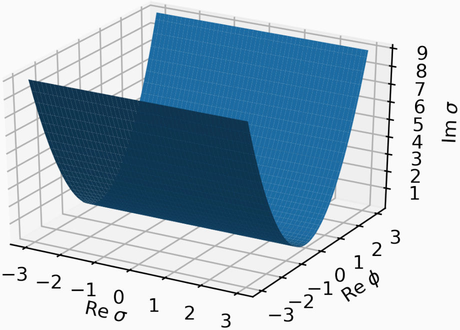
Calculate $\int \mathrm{d}x \; e^{ix^2+\lambda ix^4}$ two ways!





The Integration Contour

$$\int \mathrm{d}x\mathrm{d}\sigma\ e^{ix^2+i\lambda x^4} e^{-\sigma^2} = \int \mathrm{d}x\mathrm{d}\sigma\ e^{ix^2+i\lambda x^4} e^{-\left(\sigma+i\sqrt{i\lambda}x^2\right)^2}$$



Generalizing to Fields

$$\int \mathrm{d}x\ e^{\alpha x^2 + \lambda x^4} = \int \mathrm{d}\sigma\ e^{-\sigma^2} \frac{1}{\sqrt{\alpha + 2i\sqrt{\lambda}\sigma}}$$

Generalize to V sites with quadratic term xMx:

$$\int \mathrm{d}^V x\ e^{-x^T M x - \sum_i \Lambda_i x_i^4} = \int \mathrm{d}^V \sigma\ e^{-\sum \sigma^2} \frac{1}{\sqrt{\det\left(M_{ij} + 2i\sqrt{\Lambda_i}\sigma_i \delta_{ij}\right)}}$$

Insanity Check

$$S(x,y) = ix^2 + iy^2 + i\lambda x^4 + i\lambda y^4$$

Compute by direct integration, and using two auxiliary fields:

$$\int \mathrm{d}x\mathrm{d}y\; e^{-S} \stackrel{?}{=} \int \mathrm{d}\sigma_x \mathrm{d}\sigma_y e^{-\sigma_x^2-\sigma_y^2} \frac{1}{\sqrt{\left(i - 2i\sqrt{i\lambda}\sigma_x\right)\left(i - 2i\sqrt{i\lambda}\sigma_y\right)}}$$

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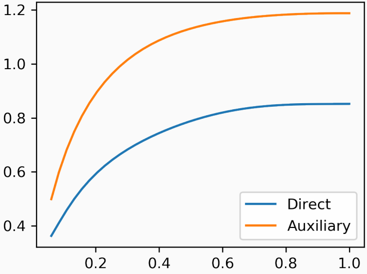


How can this be?

The partition function factorizes!

$$Z = \int \mathrm{d}x\mathrm{d}y\; e^{-S(x,y)} = \left(\int \mathrm{d}x\; e^{-ix^2 - i\lambda x^4}\right)^2$$

We've already checked the 1D case:

$$\int \mathrm{d}x\; e^{-ix^2 - i\lambda x^4} = \int \mathrm{d}\sigma\; e^{-\sigma^2} \frac{1}{\sqrt{i + 2i\sqrt{i\lambda}\sigma}}$$

What could go wrong?

$$\int \mathrm{d}\sigma_x \mathrm{d}\sigma_y\; e^{-\sigma_x^2 - \sigma_y^2} \frac{1}{\sqrt{i + 2i\sqrt{i\lambda}\sigma_x}} \frac{1}{\sqrt{i + 2i\sqrt{i\lambda}\sigma_y}}$$

Branches

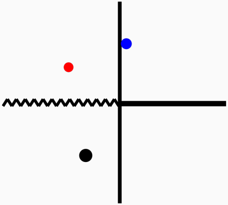
$$\int d\sigma_x d\sigma_y \; e^{-\sigma_x^2-\sigma_y^2} \frac{1}{\sqrt{i+2i\sqrt{i\lambda}\sigma_x}} \frac{1}{\sqrt{i+2i\sqrt{i\lambda}\sigma_y}}$$

is not the same as

$$\int d\sigma_x d\sigma_y \; e^{-\sigma_x^2-\sigma_y^2} \frac{1}{\sqrt{\left(i+2i\sqrt{i\lambda}\sigma_x\right)\left(i+2i\sqrt{i\lambda}\sigma_y\right)}}$$

In general:

$$\sqrt{\sigma_x \sigma_y} \neq \sqrt{\sigma_x}\sqrt{\sigma_y}$$



$\sqrt{i} = e^{i\pi/4}$ and $e^{3i\pi/4} = e^{3i\pi/8}$

$$\sqrt{i}\sqrt{e^{3i\pi/4}} = -\sqrt{(i)(e^{3i\pi/4})}$$

More Generally

$$Z(\alpha, \beta) = \int \mathrm{d}x \int \mathrm{d}\sigma \; e^{-\sigma^2} e^{\alpha x^2 + \beta^2 \sigma x^2}$$

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We can't switch the order of the integrals!

$$\int \mathrm{d}x \; e^{\alpha x^2 + 2i\beta\sigma x^2}$$

Nevertheless, Z is an analytic function of α and β. For imaginary β:

$$\tilde{Z}(\alpha, \beta) = \int \mathrm{d}\sigma \; e^{-\sigma^2} \frac{1}{\sqrt{\alpha + i\beta\sigma}}$$

But (picking the right branch) Z and $\tilde{Z}$ are both analytic!

$$Z = \tilde{Z}$$

Analytic Continuation

$$\int \mathrm{d}^V x \; e^{-x^T M x - \sum_i \Lambda_i x_i^4} = \int \mathrm{d}^V \sigma \; e^{-\sum \sigma^2} \frac{1}{\sqrt{\det\left(M_{ij} + 2i\sqrt{\Lambda_i}\sigma_i \delta_{ij}\right)}}$$

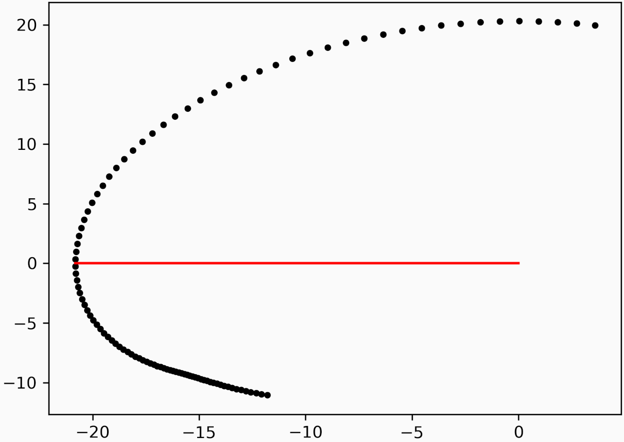
where the branch of the square root is determined by analytic continuation.

$$Z(\theta) = \operatorname{Tr} \exp\left(-\beta H\right) \exp\left(e^{i\theta} H t\right) \exp\left(e^{-i\theta} H t\right)$$

Then M and Λ are functions of θ, and we define $f(\theta)$ to be analytic.

$$f(\theta) = \frac{1}{\sqrt{\det\left(M_{ij}(\theta) + 2i\sqrt{\Lambda_i(\theta)}\sigma_i \delta_{ij}\right)}}$$

Branch Tracking



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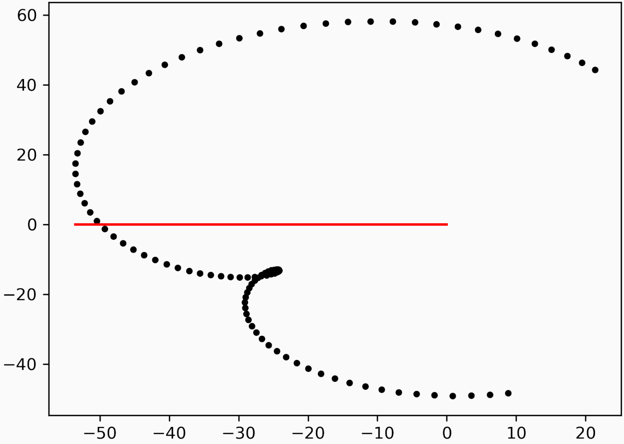
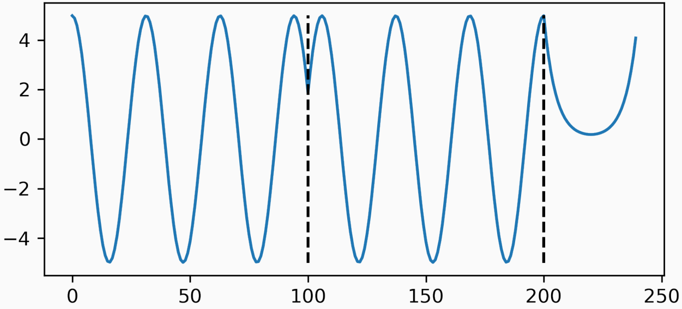


Correlators

Again, the same as with fermions.

$$\left\langle \bar{\psi}_a \psi_b \right\rangle = Z^{-1} \int \mathcal{D}\sigma e^{-S_{\text{eff}}[\sigma]} D^{-1}_{ab}$$

$$\langle \phi_a \phi_b \rangle = Z^{-1} \int \mathcal{D}\sigma e^{-S_{\text{eff}}[\sigma]} M^{-1}_{ab}$$



When Sampling

1. Sample σ.
2. Compute determinant for ϕ.
3. Accept/reject on $|\det M|^{-1/2}$, ignoring the branch cut.

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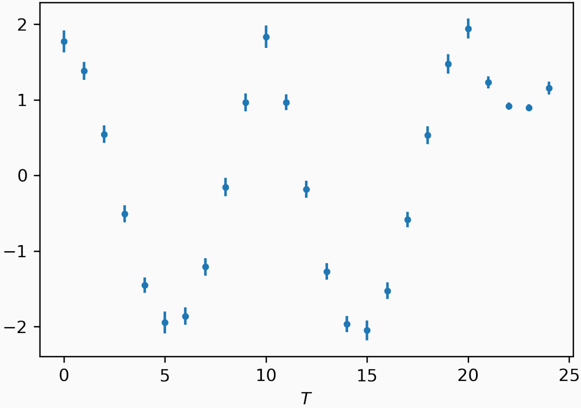
When Measuring

1. Compute sign of $\sqrt{\det M}$ via analytic continuation.
2. Reweight with phase of $\sqrt{\det M}$.
3. Compute correlator as $\langle \phi_a \phi_b \rangle \sim M^{-1}_{ab}$.

Measuring is the slow part!

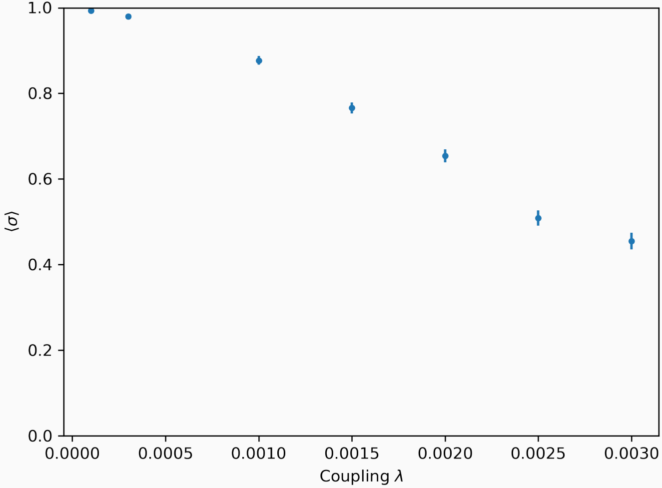
Anharmonic Oscillator

$$S_E = \frac{m^2}{2}\phi^2 + \lambda\phi^4$$



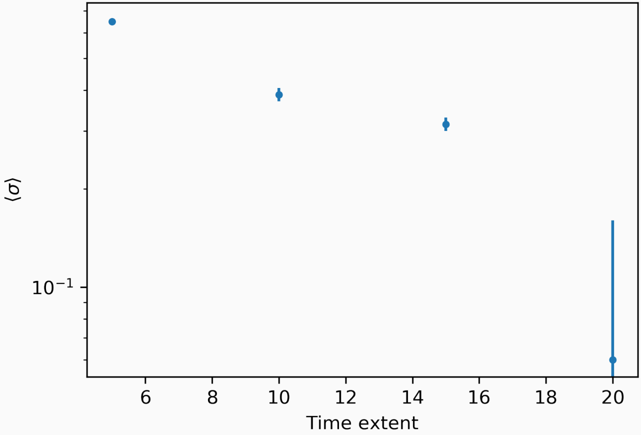
$$\beta = 5,\ m^2 = 0.3,\ \lambda = 0.001$$

Scaling — Coupling



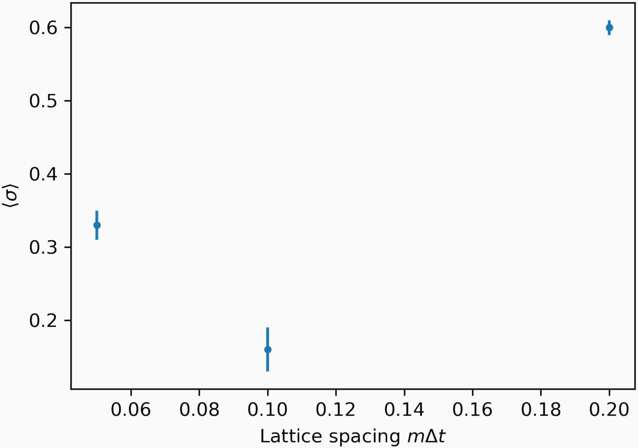
Calculated with $\beta = 5,\ T = 5,\ m^2 = 0.2$.

Scaling — Time Extent



$$\beta = 5,\ m^2 = 0.2,\ \lambda = 0.002$$

Scaling — Continuum



$$\beta = 5,\ T = 5,\ \lambda/m^2 = 0.01$$

Obstacles and Outlook

- Stability of determinant
- Push to $1+1$
- Gauge theories

Broader Lesson?

Arrange configuration space to have gaussian 'slices'.

Outline